Mathematics Methods
Unit 3

Binomial distribution

1. Bernoulli trial

Bernuoli trial or binomial trial propeties (charactesristics):

- Each of the trials has only two outcomes: success,p or failure,q
- Trials are independent of each other (outcomes of previous trial has no influence on the outcome of the following trial
- Trials are discrete random variable

Bernuoli trial can be represented by:

$$p + q = 1$$

p: probability of success
q: probability of failure

2. Bernoulli random variable

(a) Mean/ expected value

Formula:

$$\mu = p$$

Derivation of formula:

$$E(X) = \sum x \times P(X = x)$$
$$= P(X = 0) + P(X = 1)$$
$$= p(1) + q(0)$$
$$= p$$

Example 1:
A bag contains two types of cards: black card and gold card. There are one gold card and three black cards. The random variable x is defined as the number of black card(s) drawn. There is only a chance in which the cards can be drawn. Determine the mean of the distribution.

x	0	1
$P(X = x)$	$\frac{1}{4}$	$\frac{3}{4}$

$$\mu = \frac{3}{4}$$

Example 2:
Given that $Z = 1$ when a jocker card is drawn while $Z = 0$ for all other cards drawn. Calculate the expected value of Z in a traditional deck of cards.

A traditional deck of cards has 54 cards.

$$\mu = \frac{2}{54} = \frac{1}{27}$$

(b) Variance

Formula:

$$\sigma^2 = pq$$

Derivation of formula:

$$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2 \\ &= p - p^2 \\ &= p(1-p) \\ &= pq \end{aligned}$$

$$\begin{aligned} E(X^2) &= \sum x^2 \times P(X=x) \\ &= x^2\,P(X=1) + x^2\,P(X=0) \\ &= 1^2 P(X=0) + 0^2 P(X=1) \\ &= p \end{aligned}$$

$$[E(X)]^2 = p^2$$

Example 1:
Calculate the variance of a distribution if the probability of success is 0.2 while the probability of failure is 0.8.

$$\begin{aligned} \sigma^2 &= (0.2)(0.8) \\ &= 0.16 \end{aligned}$$

Example 2:
Based on the previous statistics, a school has 4% of students scoring straight A's in the year 11 examination. The school principal made a forecast that the forecast students obtaining straight A's is the same as the previous year.
$Y = 1$ is defined when a student falls under the straight A's category while $Y = 0$ is defined when a student falls under other categories other than straight A's. Find the variance of the distribution.

$$\begin{aligned} \sigma^2 &= (0.04)(0.96) \\ &= 0.0384 \end{aligned}$$

(c) Standard deviation

Formula:

$$\sigma = \sqrt{pq}$$

Derivation of formula:

$$\begin{aligned} Std(X) &= \sqrt{Var(X)} \\ &= \sqrt{pq} \end{aligned}$$

Example:
The probability distribution of an event is shown in the table below:

x	0	1
$P(X = x)$	0.4	0.6

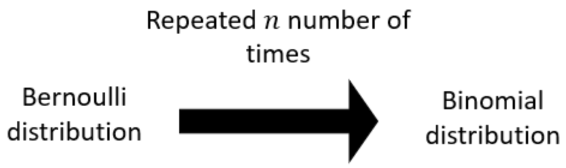
Find the standard deviation of the distribution.

$$\sigma = \sqrt{0.4(0.6)}$$
$$= \sqrt{0.24}$$
$$= 0.4899$$

3. Binomial distribution

Definition: Binomial distribution is a type of discrete random variable (specific), counting the probability of an event over a fixed number of trials.

How binomial distribution is formed?



Binomial distribution can be denoted by:

$$X \sim B(n, p)$$

n: number of trials/ repetition
p: probability of success

Binomial distribution properties (characteristics):

- Has only two outcomes (success,p or failure,q)
- Probability is constant for success and failure
- It is repeated for a number of times/ trials
- Trials are independent of one and another

(a) Probability

Parameters

$$X \sim B(n, p)$$

n: number of repitition
p: probability of success

Formula

$$P(X = r) = {}^{n}C_{r} p^{r} q^{n-r}$$

Where:
X: discrete random variable in a binomial distribution
n: number of trial
r: number of success
$n - r$: number of failure
p: probability of success
q: probability of failure

Example 1:
Given that $X \sim B(8, 0.5)$, find:

$P(X > 2)$	$P(X > 2) = 1 - P(X=0) + P(X=1) + P(X=2)$ $= 1 - 0.1445$ $= 0.8555$
$P(X \leq 2)$	$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$ $= {}^8C_0(0.5)^0(0.5)^8 + {}^8C_1(0.5)^1(0.5)^7 + {}^8C_2(0.5)^2(0.5)^6$ $= 0.1445$
$P(X \leq 5 \mid X \geq 1)$	$P(X \leq 5 \mid X \geq 1) = \frac{P(X \leq 5 \cap X \geq 1)}{P(X \geq 1)}$ $= \frac{P(X=1) + P(X=2) + \ldots + P(X=5)}{1 - P(X=0)}$ $= \frac{1 - P(X=0) - P(X=6) - P(X=7) - P(X=8)}{1 - 0.003906}$ $= \frac{0.851563}{0.996094}$ $= 0.855$

Example 2:
In a survey, 20% of the students likes mathematics. If a random sample of 4 students are choosen, find the probabilty that two of them like mathematics.

$$P(X=2) = {}^4C_2(0.2)^2(0.8)^{4-2}$$
$$= 0.1536$$

Example 3:
Azri is salesman. It is known that the probability of getting a potential customer in sales is 0.05. What is the least number of calls that must be made to ensure that the probability of making at least 2 sales is more than 90%?

$p = 0.05, \quad q = 0.95$

$$P(X \geq 2) > \frac{90}{100}$$
$$1 - \mathrm{P}(\mathrm{X}=0) - \mathrm{P}(\mathrm{X}=1) > 0.9$$
$$\mathrm{P}(\mathrm{X}=0) + \mathrm{P}(\mathrm{X}=1) < 0.1$$
$$nC0\,(0.05)^0(0.95)^n + nC1(0.05)^1(0.95)^{n-1} < 0$$
$$0.95^n + 0.05n\,(0.95)^{n-1} < 0.1 \text{ (solve with calculator)}$$
$$n < 76.3367$$
$$n = 75$$

Example 4:
A random variable is binomially distributed. It's variance is 2 while mean is 6. Find $P(X=3)$.

$$npq = 2$$
$$np = 6$$
$$q = \frac{2}{6}, \quad p = \frac{4}{6}$$
$$n = 9$$

$$P(X=3) = {}^9C_3\left(\frac{4}{6}\right)^3\left(\frac{2}{6}\right)^6$$
$$= 0.0341$$

(b) Mean

Formula:

$$\mu = np$$

Derivation of formula:

$$E(X) = \sum_{n} x \times P(X = x)$$

$$= \sum_{x=0}^{n} x \times P(X = x)$$

$$= \sum_{x=0}^{n} x \times {}^{n}C_r p^r q^{n-r}$$

$$= \sum_{x=0}^{n} x \times \frac{n!}{x!\,(n-x)!} \times p^x(1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!\,(n-x)!} \times p^x(1-p)^{n-x}$$

$Let\ R = x - 1\ \&\ m = n - 1,$

$$= \sum_{R=0}^{m} \frac{(m+1)!}{R!\,(m-R)!} \times p^{R+1}(1-p)^{m-R}$$

$$= (m+1)p \sum_{R=0}^{m} \frac{m!}{R!\,(m-R)!} \times p^{R}(1-p)^{m-R}$$

$$= np \ * \sum_{R=0}^{m} \frac{m!}{R!\,(m-R)!} \times p^{R}(1-p)^{m-R}$$

$$= np(1)$$

$$= np$$

*Binomial theorem,

$$(a+b)^m = \sum_{R=0}^{m} \frac{m!}{R!\,(m-R)!} \times a^R(b)^{m-R}$$

$Let\ a = p, \quad b = 1 - p$

$$\sum_{R=0}^{m} \frac{m!}{R!\,(m-R)!} \times p^{R}(1-p)^{m-R} = \sum_{R=0}^{m} \frac{m!}{R!\,(m-R)!} \times a^R(b)^{m-R}$$

$$= (a+b)^m$$

$$= (p + 1 - p)^m$$

$$= 1^m$$

$$= 1$$

Example 1:
Kobauro Ltd. predicts that out of the monthly production batch of motherboards, 2% are defective due to various reasons. Find the expected number of defective motherboards in a sample of 400 motherboards.

$\mu = np$
$= (400)(0.02)$
$= 8$

Example 2:
X is a discrete random variable such that $X \sim B(n, p)$. If the value of $q = 0.1$ and the variance is 34, find the mean of distribution.

$\sigma^2 = npq$
$34 = np(0.1)$
$np = 340$

(c) Variance

Formula:

$$\sigma^2 = npq$$

Derivation of formula:

$\sigma^2 = E(X^2) - [E(X)]^2$
$= E(X^2) - E(X) + E(X) - [E(X)]^2$
$= E[x(x-1)] + E(X) - [E(X)]^2$
$=* \, n(n-1)p^2 + np - n^2p^2$
$= n^2p^2 - np^2 + np - n^2p^2$
$= -np^2 + np$
$= np(1-p)$
$Since\ q = 1 - p,$
$= npq$

$$* \ E[x(x-1)] = \sum_{x=2}^{n} \frac{n!}{(x-2)!\,(n-x)!} \times p^x (1-p)^{n-x}$$

$Let\ R = x - 2\ \&\ m = n - 2,$

$$= n(n-1) \sum_{R=0}^{m} \frac{m!}{R!\,(m-R)!} \times p^{R+2} (1-p)^{m-R}$$

$$= n(n-1)p^2 \sum_{R=0}^{m} \frac{m!}{R!\,(m-R)!} \times p^{R} (1-p)^{m-R}$$

$$= n(n-1)p^2 (p + 1 - p)^m$$

$$= n(n-1)p^2$$

Example 1:
Michael who sells "economy rice" predicts that out of the daily dish cooked, 26% are left unsold. Find the variance of the leftover dishes at the end of the day if Michael cooks 25 dishes that day.

$$\begin{aligned}\sigma^2 &= npq \\ &= 25(0.26)(0.74) \\ &= 4.81\end{aligned}$$

Example 2:
Kaishan throws a biased die 20 times and the number of 2's seen is 8 times. Find the variance for the appearance of the number, 2.

$$\begin{aligned}\sigma^2 &= npq \\ &= 20\left(\frac{8}{20}\right)\left(\frac{12}{20}\right) \\ &= 4.8\end{aligned}$$

(d) Standard deviation

Formula:

$$\sigma = \sqrt{npq}$$

Derivation of formula:
(same as of variance)

$$\begin{aligned}\sigma^2 &= npq \\ &= \sqrt{npq}\end{aligned}$$

Example 1:
Given that $X \sim B(n, p)$ and the value of $q = 0.9$ while $n = 5$. Calculate the standard deviation.

$$\begin{aligned}\sigma &= \sqrt{npq} \\ &= \sqrt{5(0.9)(0.1)} \\ &= \sqrt{0.45} \\ &= 0.6708\end{aligned}$$

Example 2:
Suppose that a fair coin is flipped 30 times. Calculate the standard deviation.

$$\begin{aligned}\sigma &= \sqrt{npq} \\ &= \sqrt{30(0.5)(0.5)} \\ &= \sqrt{7.5} \\ &= 2.7386\end{aligned}$$

END